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Standard-Model Bundles on Non-Simply Connected Calabi–Yau Threefolds

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Abstract: We give a proof of the existence of $G = SU(5)$, stable holomorphic vector bundles on elliptically fibered Calabi–Yau threefolds with fundamental group $\mathbb{Z}_2$. The bundles we construct have Euler characteristic 3 and an anomaly that can be absorbed by M-theory five-branes. Such bundles provide the basis for constructing the standard model in heterotic M-theory. They are also applicable to vacua of the weakly coupled heterotic string. We explicitly present a class of three family models with gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

Keywords: Standard Model, Heterotic M-theory.

Contents

1. Introduction

Hořava–Witten theory [1] can be consistently compactified on a Calabi–Yau threefold with non-vanishing four-form "G-flux" [2]. It was shown in [3] that this reduction leads to a new low-energy limit of M-theory. This limit consists of a five-dimensional "bulk" space with $N = 1$ local supersymmetry, bounded by two, four-dimensional, $N = 1$ supersymmetric $\mathbb{Z}_2$-orbifold fixed planes. Furthermore, the theory can admit BPS fivebranes located in the bulk space, each with two spacelike directions wrapped on a holomorphic curve in the Calabi–Yau threefold [2, 4]. The worldvolumes of these wrapped fivebranes possess $N = 1$ supersymmetry. Hence, this limit of M-theory, called heterotic M-theory, provides an explicit description of a "brane" universe, derived directly from a fundamental theory.

In addition to its brane structure, heterotic M-theory can also account for much of phenomenological particle physics. In keeping with the brane context, three families of $N = 1$ supersymmetric quarks and leptons can be shown to exist naturally on one of the orbifold fixed planes, which we call the "observable" brane. These transform under either grand unified gauge groups, such as $SO(10)$ and $SU(5)$, or under the standard model gauge group,

$SU(3)_C \times SU(2)_L \times U(1)_Y$. The supersymmetric matter and the associated gauge groups arise from the higher dimensional Hořava–Witten theory as follows. Hořava–Witten theory consists of an eleven-dimensional $N = 1$ locally supersymmetric bulk space bounded by two ten-dimensional $\mathbb{Z}_2$-orbifold fixed planes exhibiting $N = 1$ supersymmetry. It was shown in [1] that anomaly cancellation requires the existence of an $N = 1$, E_8 Yang–Mills supermultiplet on each of the orbifold fixed planes. This theory is then dimensionally reduced on a Calabi–Yau threefold, Z, with a non-vanishing four-form flux as required by anomaly cancellation. On either one of the orbifold fixed planes, the result at low energy is a four-dimensional, $N = 1$ supersymmetric theory with matter and gauge group content arising from the "decomposition" of the E_8 supermultiplet under the dimensional reduction. This decomposition is entirely controlled by the vacuum structure of the E_8 gauge fields on the Calabi–Yau threefold. As discussed in [4], some subgroup $G \subseteq E_8$ of the gauge fields can be non-vanishing on Z. These "G-instantons" will preserve $N = 1$ supersymmetry as long as they satisfy the Hermitian Yang–Mills equations. However, the low energy gauge group is altered by these instantons, being spontaneously broken from E_8 to H, where $H \subseteq E_8$ is the commutant of the instanton structure group G, that is, the largest subgroup in E_8 such that $[H, G] = 0$. This mechanism, originally stated in [5], allows GUT groups, such as $SO(10)$ and $SU(5)$ (the commutants of $G = SU(4)$ and $SU(5)$ respectively), and the standard model gauge group (the commutant of $G = SU(5) \times \mathbb{Z}_2$, for example) to appear on the observable brane. Furthermore, the decomposition of the E_8 Yang–Mills supermultiplet under G into H multiplets determines the structure of matter on the observable brane.

The construction of heterotic M-theory models with grand unified gauge groups H and three families of matter is relatively straightforward, and has been discussed in [6, 7, 8, 9, 10]. Specifically, such theories require the construction of G-instantons on a simply connected elliptically fibered Calabi–Yau threefold with a zero section. Such instantons can be computed using the theorem of Donaldson [11] and Uhlenbeck–Yau [12], which relates them to polystable holomorphic vector bundles on Z, and extensions [6, 7] of the work of Friedman, Morgan and Witten [13], Donagi [14] and Bershadsky *et al.* [15], which constructs vector bundles via the method of spectral covers. However, constructing G-instantons that lead to the standard model gauge group $H = SU(3)_C \times SU(2)_L \times U(1)_Y$ and matter content is considerably more difficult. The reason is that, in addition to vector bundles with continuous structure group G, one must introduce Wilson lines into the theory [16, 17]. The existence of Wilson lines, however, requires that the Calabi–Yau threefold Z have non-trivial fundamental group. Such manifolds do not admit a zero section and the vector bundle construction of [13, 14, 15] no longer applies.

To overcome this problem, it is necessary to give a method for computing polystable, holomorphic vector bundles over a non-simply connected Calabi–Yau threefold. A major step in this direction was taken in [18], where a spectral cover formalism applicable to torus fibrations without zero section was presented. (Similar constructions were considered in [19]). The torus fibrations, Z, were constructed as a quotient, $Z = X/\tau_X$, where X is a simply connected Calabi–Yau threefold with two sections and τ_X is a freely acting involution inter-

changing the sections. Finding phenomenologically acceptable heterotic M-theory vacua in this context amounts to finding τ_X-invariant stable vector bundles on X satisfying certain conditions (anomaly cancellation and 3-families). These conditions involve only the charges of the corresponding vacua, which mathematically are encoded in the Chern classes. The τ_X-invariance imposes non-trivial restrictions on the charges. In [18] we exhibited an infinite collection of τ_X-invariant admissible charges and showed that each of those is realized by a non-empty family of vector bundles on X. The next step is to prove the existence of actual τ_X-invariant vacua in such families. This is the subject of the present work.

Such a proof is considerably more difficult than showing the invariance of the Chern classes alone. Rather than consider the bundles presented in [18], for various technical reasons, it is expedient to consider a different class of vector bundles over a Calabi–Yau threefold with $\pi_1(Z) = \mathbb{Z}_2$. These bundles also lead to the standard model, but have a structure that lends itself more easily to discussing invariance under $\mathbb{Z}_2$. In this paper, we explicitly present polystable, holomorphic vector bundles of this type and prove that they are $\mathbb{Z}_2$-invariant. This is a fundamental step in demonstrating that the standard model can arise in heterotic M-theory. This construction, and the proof of the $\mathbb{Z}_2$ invariance, is necessarily of a technical nature but is sufficiently important for phenomenological physics that we synopsize it in this paper. The complete proof, with all mathematical details, is presented in a companion papers [21].

The types of invariant bundles discussed here can be extended to larger freely acting automorphism groups of X, such as $\mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and so on. We will show in a future paper that standard-like models arising from $SO(10)$ and other grand unified groups can be constructed with the possibility of suppressed nucleon decay. Finally, we would like to emphasize that, although the vector bundles discussed in [6, 7, 18] and this paper are within the context of heterotic M-theory, these bundles are equally applicable to the construction of new, phenomenologically interesting vacua in the weakly coupled heterotic string.

2. Outline of the Construction

Let us start by summarizing the problem we have to solve. Recall from the introduction that to construct an $N = 1$ vacuum in four-dimensions, we must compactify on a Calabi–Yau threefold Z and choose E_8 gauge fields in a $G \subseteq E_8$ polystable holomorphic vector bundle $\mathcal{V}$ on Z.

We would like to construct models with the standard model gauge group and three families of charged matter. In this paper, we will break the gauge group through an $SU(5)$ GUT group. We have

$$E_8 \stackrel{\text{bundle } \mathcal{V}}{\longrightarrow} SU(5) \stackrel{\mathbb{Z}_2 \text{ Wilson line}}{\longrightarrow} SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{2.1}$$

By choosing an $SU(5)$ vector bundle $\mathcal{V}$ on the Calabi–Yau manifold Z, we break the preserved gauge group to the commutant $SU(5)$. By choosing a $\mathbb{Z}_2$ Wilson line, we can then further break this down to the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

We recall that there are two additional conditions [6, 7]. First, the requirement of three families translates into a condition on the third Chern class

$$N_{\text{gen}} = \frac{1}{2} c_3(\mathcal{V}) = 3. \tag{2.2}$$

Second, in order to cancel anomalies the orbifold planes and the fivebranes must be sources of G-flux. The condition that the net charge vanishes becomes

$$c_2(\mathcal{V}) + [W] = c_2(TZ), \tag{2.3}$$

where $[W]$ is the total cohomology class of the holomorphic curves on which the fivebranes are wrapped. To describe physical branes, the class $[W]$ must be effective.

In general, then, we need to satisfy the following conditions

Supersymmetry: Z is a Calabi–Yau threefold. $\mathcal{V}$ is a polystable, holomorphic $SU(5)$ bundle,

Wilson Line: Z has, at least, $\pi_1(Z) = \mathbb{Z}_2$,

Anomaly Cancellation: $c_2(TZ) - c_2(\mathcal{V})$ must be an effective class in Z,

Three Families: $c_3(\mathcal{V}) = 6$,

in order to have a realistic model.

The simplest way to construct a suitable Calabi–Yau threefold Z is as a quotient [16]. Let X be a smooth Calabi–Yau threefold with trivial fundamental group. Suppose, in addition, we have an involution,

$$\tau_X : X \to X, \tag{2.4}$$

with $\tau_X^2 = \text{id}$, which is freely acting, that is, has no fixed points, and which preserves the holomorphic 3-form. The quotient space Z formed by identifying points related by the involution,

$$Z = X/\tau_X, \tag{2.5}$$

is then a smooth Calabi–Yau threefold with $\pi_1(Z) = \mathbb{Z}_2$.

Rather than construct the bundle $\mathcal{V}$ directly on the quotient $Z = X/\tau_X$, it is generally easier to construct it from a bundle V on X. If V is τ_X-invariant, that is

$$\tau_X^* V \cong V, \tag{2.6}$$

then, as long as V is stable, it will descend to a bundle $\mathcal{V}$ on Z.

We will construct the bundle V via the "spectral cover" construction [13, 14, 15] which, essentially, uses T-duality to describe V in terms of simpler T-dual data. However, this puts a constraint on the Calabi–Yau manifold X. It requires that X is elliptically fibered over a

two-complex-dimensional base. This means that at each point on the base, there is a torus fiber on which one can perform the T-duality. We should note also that, in general, the spectral construction gives a $U(n)$ rather than an $SU(n)$ bundle. Thus, we will need the additional condition that $c_1(V) = 0$ in order to ensure that V is an $SU(5)$ bundle.

In summary then, if Z is a quotient manifold and we build the bundle via the spectral construction, we must satisfy the following conditions

- **(Z2)** X is a smooth elliptically fibered Calabi–Yau threefold admitting a freely-acting involution $\tau_X : X \to X$,
- **(S)** V is a stable rank-five vector bundle on X,
- **(I)** V is τ_X-invariant,
- **(C1)** $c_1(V) = 0$,
- **(C2)** $c_2(TX) - c_2(V)$ is effective,
- **(C3)** $c_3(V) = 12$.

Note that the final three-family condition is now $c_3(V) = 12$ because, under the quotient, we have $c_3(\mathcal{V}) = \frac{1}{2}c_3(V)$.

Our problem, then, is to find solutions to the conditions **(Z2)**–**(C3)**. The procedure, which is summarized in Figure 1, will be as follows. First, in Section 3, we construct a large family of elliptically fibered Calabi–Yau manifolds X satisfying the involution condition **(Z2)**. Second, in Section 4, we construct a large family of bundles V on X satisfying the invariance condition **(I)**. This is the most difficult part of the construction. Finally, in Section 5, we reduce the stability condition **(S)** and the conditions on the Chern classes **(C1)**–**(C3)** to numerical conditions on the parameters defining V. A class of solutions to these conditions is then given in Section 6.

In order to simplify the construction of τ_X-invariant bundles, we specialize to Calabi–Yau manifolds X built from a particular type of complex two-fold base B known as a "rational elliptic surface" or a "dP_9". These are surfaces which are themselves elliptically fibered. In order to construct an involution τ_X on X, we need to understand the action of involutions τ_B on the surfaces B. The structure of B and the special class of B admitting τ_B, are discussed in Sections 3.1 and 3.2.

In order to find solutions to the numerical conditions, we must actually specialize further and consider only those rational elliptic surfaces where some of the elliptic fibers of B split. This introduces new effective classes on B and, hence, new freedom in constructing V. This four-dimensional sub-family of surfaces is described in Section 3.3. The manifold X is then constructed as the "fiber product" of two surfaces B and B'. (Note that Calabi–Yau manifolds of this type were also considered in [19].) The involution τ_X is similarly built from involutions τ_B and τ'_B on each of the two surfaces. This is described in Section 3.4.

Turning to the construction of V, we may take as a first approximation to V a pullback to X of a bundle W on B. The point is that on B, it is comparatively easy to find those

Construct a large family of elliptically fibered Calabi–Yau threefolds X with a freely acting involution τ_X to satisfy the condition **(Z2)** (3)

1. Construct a five-dimensional class of rational elliptic surfaces B with an involution τ_B. (3.1), (3.2)
2. Specialize to a class of (B, τ_B) with split fibers. (3.3)
3. Construct (X, τ_X) as fiber product $B \times_{\mathbb{P}^1} B'$ of special rational elliptic surfaces. (3.4)

Construct a family of τ_X-invariant vector bundles V for each (X, τ_X) via the spectral construction, thus satisfying the condition **(I)** (4)

4. Construct a class of τ_X-invariant, rank-r bundles V_r as pullbacks of τ_B-invariant bundles on B, modified by Hecke transforms . (4.2)
5. Construct V as an extension of V_2 and V_3 bundles. (4.3)

Impose the conditions **(S)**, **(C1)**, **(C2)** and **(C3)**, which become a set of numerical conditions on the parameters defining V. (5), (6)

Figure 1: Flow diagram of the construction. The numbers refer to the relevant section.

bundles which are invariant under the corresponding involution τ_B, a problem which is harder to solve on the full manifold X. However, bundles of this form are not general enough to satisfy all the numerical conditions. One is forced to make two generalizations. First, one modifies the bundle by means of "Hecke transforms" which, roughly speaking, modify the bundle over surfaces in the Calabi–Yau threefold. Here, it is the fact that the fibers of B and B' split which provides new surfaces in X, and the additional freedom to make Hecke transforms over these surfaces. This construction is described in Section 4.2.

Even with the additional freedom from Hecke transforms, the bundles one constructs, V_r for general rank r, cannot satisfy both **(C1)** and **(C3)**. This leads to the second generalization discussed in Section 4.3: the final bundle V is actually built as the "extension" of a rank-two bundle V_2 and a rank-three bundle V_3, each built using the construction just described. This means that V is a sort of "twisted" sum of V_2 and V_3.

Finally, we show in Section 6 that there is then a large class of solutions to the numerical conditions. For the example we give, the solution has four independent parameters. Other classes of examples also exist. Even given the constraints, we find that there is still a large freedom in the examples.

3. A Family of (X, τ_X)

Our first goal is to construct a large family of elliptically fibered Calabi–Yau threefolds $\pi : X \to B'$ with a freely acting involution τ_X. Some explicit examples were described in [18]. Here, we will consider a more general class of involutions but specialize to the case where the base B' of the fibration is a rational elliptic surface. This will lead to a large class of examples for which the demonstration of the invariance of the vector bundles is particularly simple.

Let us recall some properties of the construction in [18]. First, the involution τ_X will necessarily induce some involution $\tau_{B'}$ of the base B'. Let Δ be the discriminant of the fibration π. Since X is a Calabi–Yau manifold, Δ is a section of K_B^{-12}. A necessary condition for τ_X to be freely acting is that the set of fixed points of $\tau_{B'}$ be disjoint from the discriminant locus of π, that is

$$\{\text{fixed points of } \tau'_B\} \cap \{\Delta = 0\} = \varnothing. \tag{3.1}$$

Consequently, the first step in the construction of (X, τ_X) is to find suitable base pairs $(B', \tau_{B'})$. Much of this section will thus concentrate on the construction of involutions on rational elliptic surfaces.

As discussed in section 3.4, the fact that the base B' is itself an elliptic fibration, $\beta' : B' \to \mathbb{P}^1$, will allow us to construct X as the fiber product of a pair of rational elliptic surfaces B and B' over a common $\mathbb{P}^1$. Thus, in this case, simply by understanding involutions on rational elliptic surfaces we will be able to construct involutions on X. Such constructions were first considered in [22]. In fact, exactly the four dimensional subfamily of rational elliptic surfaces described in section 3.3 below happened to appear, in a different context, as an example in [22, Section 9].

3.1 Rational Elliptic Surfaces

A rational elliptic surface B is a two-dimensional complex manifold which is a fibration of elliptic curves over a sphere base $\mathbb{P}^1$

$$\beta : B \to \mathbb{P}^1. \tag{3.2}$$

It can be described as the blow-up of the projective plane $\mathbb{P}^2$ as follows. Recall that an elliptic curve, which is topologically a torus, corresponds to a cubic curve in $\mathbb{P}^2$. Consider a one-parameter family or "pencil" of cubics where each curve passes through nine fixed points $A_1, \ldots, A_9$ in $\mathbb{P}^2$. The general surface B is then the blow-up of $\mathbb{P}^2$ at the nine points and the elliptic fibers of B just correspond to the cubic curves. Under mild general position requirements, each subset of eight of the points determines the pencil of cubics and, hence, the ninth point. This implies that the rational elliptic surfaces depend on eight complex parameters. (Fixing eight points in $\mathbb{P}^2$ requires 16 parameters. However, since only the relative positions matter, we must subtract the dimension $\dim \mathbb{P}\mathrm{GL}(3, \mathbb{C}) = 8$ of the automorphism group of $\mathbb{P}^2$, leaving eight parameters.) Although not true del Pezzo surfaces, rational elliptic

surfaces are sometimes referred to as dP$_9$. (A del Pezzo surface dP_n is obtained by blowing up $n \leq 8$ points in $\mathbb{P}_2$.)

Let $e_1, \ldots, e_9$ be the classes of the exceptional divisors of B corresponding to the nine blown-up points A_i. An additional divisor l comes from the class of a line in $\mathbb{P}^2$. It is easy to see that these provide an independent basis for the cohomology of B, such that

$$H^2(B, \mathbb{Z}) = \mathbb{Z}l \oplus \left(\oplus_{i=1}^9 \mathbb{Z}e_i\right), \tag{3.3}$$

and, furthermore, the intersections between classes are given by $l^2 = 1$, $l \cdot e_i = 0$ and $e_i \cdot e_j = -\delta_{ij}$. In addition, the anti-canonical class of B is equal to the class of the elliptic fiber

$$-K_B = c_1(B) = 3l - \sum_{i=1}^{9} e_i. \tag{3.4}$$

which we denote by f. Finally, we note that since each exceptional curve $e_1, \ldots, e_9$ intersects the fiber at one point, these are all sections of the fibration. In this paper, we will always identify one section $e : \mathbb{P}^1 \to B$ as the zero section. Without loss of generality we can take $e = e_9$. Fixing the zero section determines a group law generating the translational symmetries of each smooth (toroidal) fiber of B.

3.2 Involutions of Rational Elliptic Surfaces

Let us now consider involutions of B. Given the group action on the fibers, one can define a natural involution $(-1)_B : B \to B$ of any rational elliptic surface as the extension to all of B of

$$(-1)_B(x) = -x, \tag{3.5}$$

where x is any point on a smooth fiber. This is the usual inversion symmetry of the torus so that, on any smooth fiber, the action of $(-1)_B$ will have four fixed points. In particular, one notes that this involution leaves the zero section invariant

$$(-1)_B(e) = e. \tag{3.6}$$

Recall the requirement (3.1) that the fixed points of τ_B be disjoint from the locus $\Delta = 0$ of the discriminant, which is a section of $K_B^{-12} = \mathcal{O}_B(12f)$. Clearly, this locus intersects the zero section 12 times, and, hence, is not disjoint from the set of fixed points of $(-1)_B$. Thus, $(-1)_B$ is unsatisfactory for construction of a freely acting τ_X.

Instead, we must specialize the rational elliptic surface to a family that admits additional involutions. Since B is elliptically fibered, our discussion will be much like the discussion in [18] for elliptically fibered threefolds, though with one generalization.

First note that any involution τ_B will induce an involution $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ in the base. Assuming that $\tau_{\mathbb{P}^1}$ does not act as the identity[1], any $\tau_{\mathbb{P}^1}$ will have two fixed points, which we

[1]Note that there is a whole second family of rational elliptic surfaces with τ_B built on $\tau_{\mathbb{P}^1} = \mathrm{id}_{\mathbb{P}^1}$. We will not discuss them here, but these too could be used to find suitable (X, τ_X) for constructing particle physics vacua, very much along the lines of this paper.

will denote as $0, \infty \in \mathbb{P}^1$ and which uniquely determine the involution. Let us fix a particular $\tau_{\mathbb{P}^1}$. One can then show that any τ_B satisfying $\tau_{\mathbb{P}^1} \circ \beta = \beta \circ \tau_B$ can be built out of a pair of objects (α_B, ζ). First, one needs an involution α_B which is a lift of $\tau_{\mathbb{P}^1}$ and so satisfies $\tau_{\mathbb{P}^1} \circ \beta = \beta \circ \alpha_B$. We also assume it leaves the zero section invariant

$$\alpha_B(e) = e. \tag{3.7}$$

Second, one needs a section ζ of β satisfying

$$\alpha_B(\zeta) = (-1)_B(\zeta). \tag{3.8}$$

The involution τ_B is then given by

$$\tau_B = t_\zeta \circ \alpha_B, \tag{3.9}$$

where t_ζ is the translation of the elliptic fibers defined by the section ζ

$$t_\zeta(x) = x + \zeta(p), \tag{3.10}$$

where x is any element of a smooth fiber $\pi^{-1}(p)$ over a point $p \in \mathbb{P}^1$. The conditions (3.7) and (3.9) are required to ensure $\tau_X^2 = \mathrm{id}_B$. Note that in the particular case where $\zeta = e$, then $\tau_B = \alpha_B$. Also note that this construction is a generalization of the involutions considered in [18]. In that paper, we required that $(-1)_B(\zeta) = \zeta$ so that (3.8) became $\alpha_B(\zeta) = \zeta$.

One finds [21] that there is a natural (and unique up to a twist by $(-1)_B$) way to construct α_B from the Weierstrass model of B. However, the construction does not exist for all rational elliptic surfaces B. Rather, within the eight parameter family of rational elliptic surfaces there is a five-dimensional sub-family of surfaces which admit α_B. In addition, all surfaces in this sub-family also admit a non-trivial section ζ satisfying (3.8). Rather than discuss the details of the construction, which can be found in [21], let us simply summarize the fixed point structure of α_B and τ_B.

Since the fixed points of $\tau_{\mathbb{P}^1}$ are at 0 and ∞, any fixed points of an involution of B lifting $\tau_{\mathbb{P}^1}$ must lie in the fibers $f_0 = \beta^{-1}(0)$ and $f_\infty = \beta^{-1}(\infty)$. First consider α_B. One can show that it acts as the identity on one of the fibers, say f_0 and as (-1) on f_∞. Thus, the whole fiber f_0 is fixed pointwise under α_B, as are four fixed points in f_∞. This is shown schematically in Figure 2. It might appear odd that this action treats f_0 and f_∞ asymmetrically. The point is that the Weierstrass model naturally defines two involutions preserving the zero section, α_B and $\alpha_B \circ (-1)_B$. Under the latter involution, the fixed point structure of f_0 and f_∞ is reversed.

Let us now turn to τ_B. First we note that the condition (3.8) implies that on f_0 we have $\zeta(0) = -\zeta(0)$ (that is, it is one of the four fixed points of the (-1) involution on f_0), while there is no such condition on f_∞. Provided ζ is not the zero section, translation by ζ will thus remove all the fixed points on f_0. The α_B-fixed points on f_∞ are simply translated by $\zeta(\infty)/2$. Thus τ_B has four fixed points as shown in Figure 2.

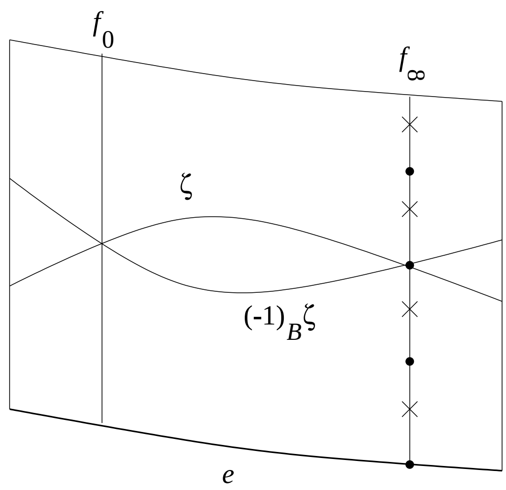

Figure 2: A general rational elliptic surface B admitting α_B.

Clearly, in general, both α_B and τ_B satisfy the condition (3.1). However, as we will discuss in section 3.4 below, it is easy to show that only τ_B leads to a freely acting τ_X. Nonetheless, it would appear that we have solved our problem, finding a five-dimensional family of (B, τ_B) suitable for building the threefold X.

3.3 Special Rational Elliptic Surfaces

It turns out that the sub-family of surfaces described in the previous section is not quite suitable for our purposes. General rational elliptic surfaces, including generic members of the sub-family, have 12 singular I_1 elliptic fibers where the torus pinches to a sphere. In constructing τ_X-invariant bundles on X, we will find that it is important for B (and hence X) to have a richer cohomology structure. In particular, we will require that B has some I_2 fibers where the torus splits into a pair of spheres. If the split fiber is not to lie over a fixed point of $\tau_{\mathbb{P}^1}$, we see that we actually need at least one pair of I_2 fibers. These will lie above a pair of points p_1 and p_2 in $\mathbb{P}^1$ exchanged by $\tau_{\mathbb{P}^1}$.

Thus the actual rational elliptic surfaces we will use in constructing X are a special four-dimensional sub-family of the family described in the previous section, with, generically, a pair of I_2 fibers and 8 I_1 fibers. These surfaces admit α_B and τ_B exactly as above, and, in particular, the fixed points of τ_B remain four points in f_∞ (see Figure 3).

Regarded as a blow-up of $\mathbb{P}^2$, the special features of this sub-family translate into special position requirements on the nine blow-up points. This is described explicitly in [21]. Again, rather than discuss the details of the construction, let us simply note the action of the τ_B involution. We have already discussed the fixed points of τ_B. What remains is to identify the cohomology classes and the induced action of τ_B on them.

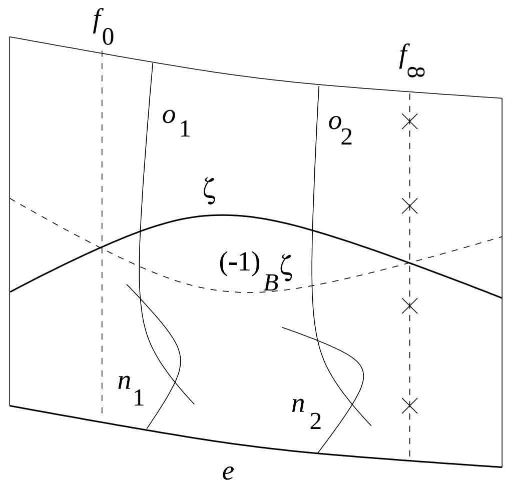

Figure 3: A special rational elliptic surface B.

	τ_B^*
$e_1 = \zeta$	e_9
e_j $(j = 2, 3)$	$f - e_j + e_1 + e_9$
e_i $(i = 4, 5, 6)$	$f - l + e_i + e_1 + e_7 + e_9$
e_7	$l - e_2 - e_3$
e_8	$f - l + e_1 + e_7 + e_8 + e_9$
$e_9 = e$	e_1
l	$2f + 2(e_1 + e_9) - (e_2 + e_3) + e_7$
f	f

Table 1: Action of τ_B^* on $H^2(B, \mathbb{Z})$

Let the I_2 fibers be f_1 and f_2. Then each fiber is a union of spheres

$$f_1 = n_1 \cup o_1, \qquad f_2 = n_2 \cup o_2. \tag{3.11}$$

As in the general case (3.3), the cohomology of B can be described in terms of nine exceptional divisors e_i and the pre-image l of the line in $\mathbb{P}^2$. As usual, we will identify $e = e_9$ as the zero section. In addition, we can identify $\zeta = e_1$ as the section defining τ_B. The explicit construction then identifies the new effective classes n_i and o_i as follows

$$\begin{aligned} n_1 &= e_8 - e_9, \\ o_1 &= 3l - e_1 - \cdots - e_7 - 2e_8 = f - n_1, \\ n_2 &= l - e_7 - e_8 - e_9, \\ o_2 &= 2l - e_1 - \cdots - e_6 = f - n_2. \end{aligned} \tag{3.12}$$

Under $\tau_{\mathbb{P}^1}$ the reducible I_2 fibers f_1 and f_2 must be exchanged. Thus, τ_B must somehow exchange (n_1, o_1) and (n_2, o_2). Specifically, one finds

$$\begin{aligned} \tau_B^*(n_1) &= o_2, \\ \tau_B^*(o_1) &= n_2. \end{aligned} \tag{3.13}$$

The full action of τ_B^* on the cohomology is given in Table 1.

This completes the description of the family of rational elliptic surfaces and the involutions τ_B which we will use to construct (X, τ_X).

3.4 Construction of (X, τ_X)

Given our specific family of rational elliptic surfaces B, we can now describe the construction of a suitable family of elliptically fibered (X, τ_X). The fact that rational elliptic surfaces are themselves elliptically fibered, allows a particularly simple construction of X as the fiber product,

$$X = B \times_{\mathbb{P}^1} B', \tag{3.14}$$

of a pair of rational elliptic surfaces B and B'. That is to say, X fits into a commutative diagram of projections

$$\begin{tikzcd} & X \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ B \arrow[dr, "\beta"'] & & B' \arrow[dl, "\beta'"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{3.15}$$

The space X is formed by taking the union, as p varies in $\mathbb{P}^1$, of the product of the fibers $\beta^{-1}(p) \times \beta'^{-1}(p)$ above p. This is shown schematically in Figure 4.

For generic choice of B and B', X will be smooth. It is an elliptic fibration in two ways: either via π or π'. Since most of our construction will center on the elliptic fibers, we will make the somewhat unconventional choice that the primed object B' is the base of the fibration, simply to avoid cumbersome notation. Much of the structure of the fibration is inherited from the structure of the β fibration of B. For instance, the discriminant of π is the pull-back of the discriminant of β and so is a section of $\mathcal{O}(12f') = K_{B'}^{-12}$. Hence, $c_1(X) = 0$ and X is Calabi–Yau. Similarly, the zero section $\sigma : B' \to X$ of π is inherited from the zero section $e : \mathbb{P}^1 \to B$, and is given by $\sigma = e \times_{\mathbb{P}^1} B'$.

Let us now assume that B and B' are special in the sense of section 3.3. We then have involutions α_B, τ_B and $\alpha_{B'}$, $\tau_{B'}$ acting on B and B' from which we will construct τ_X. There is still some freedom in how we choose to identify the $\mathbb{P}^1$ bases of B and B'. However, since all involutions of X must induce the same involution $\tau_{\mathbb{P}^1}$, there are only two possibilities,

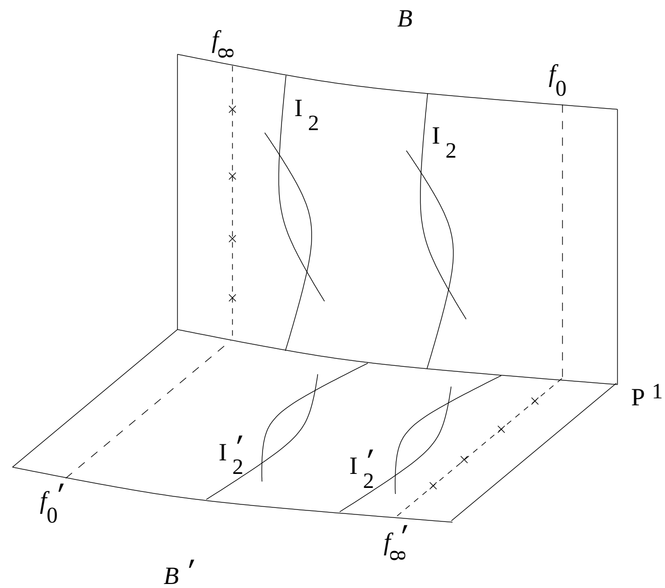

Figure 4: The structure of X.

depending on the identification of fixed points. Either we identify $0 \in \mathbb{P}^1$ with $0' \in \mathbb{P}^{1'}$ and $\infty \in \mathbb{P}^1$ with $\infty' \in \mathbb{P}^{1'}$, or 0 with ∞' and ∞ with $0'$. Suppose we make the first identification. Recall that both α_B and τ_B leave four points on f_∞ fixed. Since in the first case, both f_∞ and f'_∞ lie above the same point in $\mathbb{P}^1$, it is clear that no combinations of involutions in B and B' can be freely acting. However, with the second identification it is easy to see that the involution

$$\tau_X = \tau_B \times_{\mathbb{P}^1} \tau_{B'} \tag{3.16}$$

acts freely on X. This is because the four fixed points of τ_B and $\tau_{B'}$ live in fibers above different points in $\mathbb{P}^1$, as shown in Figure 4. Note that this would never be the case if τ_X was built from either α_B or $\alpha_{B'}$.

As for the involutions of the rational elliptic surfaces, the involution τ_X can also be built from an involution α_X preserving the zero section σ and a translation t_{ζ_X} by a second section ζ_X. (This was described in [18].) Both are built out of the corresponding structures on B, namely

$$\alpha_X = \alpha_B \times_{\mathbb{P}^1} \tau_{B'}, \tag{3.17}$$

and

$$\zeta_X = \zeta \times_{\mathbb{P}^1} B'. \tag{3.18}$$

In what follows, it will be useful to identify the classes of divisors $H^2(X,\mathbb{Z})$ on X. The fiber product structure means that

$$H^2(X,\mathbb{Z}) = \frac{H^2(B,\mathbb{Z}) \times H^2(B',\mathbb{Z})}{H^2(\mathbb{P}^1,\mathbb{Z})} \tag{3.19}$$

That is, all divisor classes are either pull-backs of classes from B or of classes from B', modulo the one relation on the fiber classes that $\pi^*(f') = \pi'^*(f)$.

Finally, we will also need the expression for $c_2(X)$ in order to solve the condition **(C2)**. In general, this is given by a curve in X. Recalling that $c_2(B) = c_2(B') = 12$ and the fibered structure of X, it is easy to show that $c_2(X)$ is given solely by some number of π and π' fibers. That is

$$c_2(X) = 12\left(f \times \mathrm{pt} + \mathrm{pt} \times f'\right), \tag{3.20}$$

where pt is the class of a point in the relevant manifold (B or B' or later X).

In summary, we have described the construction of a large class of threefolds X with freely acting τ_X. The quotient $Z = X/\tau_X$ will thus be smooth, with non-trivial $\pi_1(Z)$. We note that it is not difficult to show that Z is also a Calabi–Yau manifold, as required.

4. A Family of τ_X-Invariant Bundles V

In this section, we will describe the construction a large family of stable bundles V on X which are invariant under the involution τ_X. As in previous papers [6, 7, 8, 18], we will use the spectral construction. One new feature is that we are forced to work with a reducible spectral cover. Rather than describing the complicated behavior of the spectral sheaf at the singularities of the spectral cover, we chose to realize the resulting vector bundle as an extension of two vector bundles V_2 and V_3 each coming from more manageable spectral data. This approach is a variation of an idea of Richard Thomas [20].

The key ingredient in the spectral construction is the fact that X is elliptically fibered. This allows V to be constructed via the "Fourier–Mukai transform" which, in physics terms, is the action on the bundle of T-duality along the elliptic fibers. Recall that, with respect to the complex structure, the T-dual Calabi–Yau manifold is isomorphic to X. Formally, the Fourier–Mukai transform $\boldsymbol{FM}_X$ is then an "autoequivalence" of the "derived category" $D^b(X)$ of sheaves on X,

$$\boldsymbol{FM}_X : D^b(X) \to D^b(X). \tag{4.1}$$

Physically, we can think of a sheaf as describing a D-brane. Thus, as expected, $\boldsymbol{FM}_X$ maps one configuration of D-branes to another. It is really a little subtler. The objects in $D^b(X)$ are actually not single sheaves but complexes of sheaves. Although very important for the details of the construction (see [21]), this subtlety will not generally concern us here.

The usefulness of the Fourier–Mukai transform is that it allows one to describe V in terms of its simpler T-dual data. In particular consider a line bundle $\mathcal{N}_\Sigma$ over a smooth

surface $i_\Sigma : \Sigma \hookrightarrow X$ in X which is a finite r-fold cover of the base B'. The transform of the corresponding sheaf $i_{\Sigma *}\mathcal{N}_\Sigma$ on X,

$$V = \boldsymbol{FM}_X(i_{\Sigma *}\mathcal{N}_\Sigma), \tag{4.2}$$

is then precisely the object we want: a stable vector bundle over X of rank r. (We should note that the stability also depends on a choice of a suitable Kähler form on X.) The surface Σ is the spectral cover, $\mathcal{N}_\Sigma$ the spectral datum and the correspondence between $(\Sigma, \mathcal{N}_\Sigma)$ and V is commonly known as the spectral construction.

In order to find τ_X-invariant bundles, we would like to translate the invariance condition into a condition on $(\Sigma, \mathcal{N}_\Sigma)$. Since $\boldsymbol{FM}_X$ is invertible, we can construct the induced action of τ_X on the spectral sheaf $i_{\Sigma *}\mathcal{N}_\Sigma$

$$\boldsymbol{T}_X = \boldsymbol{FM}_X^{-1} \circ \tau_X^* \circ \boldsymbol{FM}_X. \tag{4.3}$$

The search for invariant bundles V is then reduced to finding $(\Sigma, \mathcal{N})$ such that

$$\boldsymbol{T}_X(i_{\Sigma *}\mathcal{N}_\Sigma) \cong i_{\Sigma *}\mathcal{N}_\Sigma. \tag{4.4}$$

In general, the action of $\boldsymbol{T}_X$ is extremely hard to calculate. One particular problem is that the space $\mathrm{Pic}(\Sigma)$ of line bundles $\mathcal{N}_\Sigma$ on Σ is generically not simply characterized by pullbacks of bundles from X. Instead, as we saw in [18], new divisor classes on Σ appear. Calculating $\boldsymbol{T}_X$ for the corresponding bundles is difficult. However, here, the fiber-product structure of X will come to our aid. Just as many of the properties of X were inherited from the vertical surface B, in the cases of interest, we will be able to build the Fourier–Mukai transform $\boldsymbol{FM}_X$ from the horizontal pullback by π'^* of the corresponding transform $\boldsymbol{FM}_B$ on B.

Our first step in this section is, thus, to give some results on the action of $\boldsymbol{FM}_B$ and the corresponding $\boldsymbol{T}_B$ on the special rational elliptic surfaces B. We then use these results, together with the technique of modifying bundles by "Hecke transforms," to construct a large class of τ_X-invariant bundles.

4.1 The Fourier–Mukai Transform on B and a No-Go Theorem

Since B is elliptically fibered, there is also a Fourier–Mukai action $\boldsymbol{FM}_B$ on sheaves on B. Similarly, given the involution τ_B on B, there is an induced action $\boldsymbol{T}_B = \boldsymbol{FM}_B^{-1} \circ \tau_B^* \circ \boldsymbol{FM}_B$ on sheaves on B. Let us now simply state some results for $\boldsymbol{FM}_B$ and $\boldsymbol{T}_B$. Details can be found in [21].

Our main result is the following. Let L be a line bundle on B. In general $\boldsymbol{FM}_B(L)$ will be some complex of sheaves. Nonetheless, one can show, as long as $c_1(L) \cdot o_i = 0$, that $\boldsymbol{T}_B(L)$ is actually still a line bundle. Explicitly, in terms of the divisor classes defined in section 3.3, one shows (see [21, Part I,Theorem 7.1]):

$$\boldsymbol{T}_B(L) = \tau_B^*(L) \otimes \mathcal{O}\big([c_1(L)\cdot(e-f)]f + [c_1(L)\cdot f](e-\zeta-f)\big) \otimes \mathcal{O}(e-\zeta-f). \tag{4.5}$$

Thus we see that $\boldsymbol{T}_B$ induces a complicated affine action on the space of line bundles $\mathrm{Pic}(B)$. The first two terms give the underlying linear part of the transformation, while the last term gives the constant shift.

The analogue in B of the spectral cover Σ is a smooth curve $i_C : C \hookrightarrow B$ which is a finite cover of the base $\mathbb{P}^1$. The analog of $\mathcal{N}_\Sigma$ is then a line bundle $\mathcal{N}$ on C. Consequently, we would also like to know the action of $\boldsymbol{T}_B$ on spectral sheaves $i_{C*}\mathcal{N}$. Let us assume that the bundle $\mathcal{N}$ is the pullback $\mathcal{N} = i_C^*(L)$ of a global bundle L in B (note that, in general, this is not always the case). One can then show that

$$\boldsymbol{T}_B(i_{C*}i_C^*(L)) = i_{D*}i_D^*(\boldsymbol{T}_B(L)), \tag{4.6}$$

where

$$D = \alpha_B(C) \tag{4.7}$$

is the image of C under the involution α_B. This matches the result of [18], where it was shown that the spectral cover transforms under the involution preserving the zero section, α_B, rather than the full involution, τ_B, of the manifold.

Finally, if we restrict C to be in the divisor class $re + kf$ for some integers r and f, one specifically finds, using (4.5), that

$$\boldsymbol{T}_B(i_{C*}i_C^*(L)) = i_{D*}i_D^*\left(\alpha_B^*(L) \otimes \mathcal{O}(e - \zeta - f)\right). \tag{4.8}$$

We can now use these results for $\boldsymbol{T}_B$ to show a useful no-go theorem for constructing suitable bundles on X. The most obvious simplification in the spectral construction is to ignore any new classes on the spectral cover Σ, and to assume that the line bundle $\mathcal{N}_\Sigma$ is the pullback of a global line bundle $\mathcal{L}$ on X

$$\mathcal{N}_\Sigma = i_\Sigma^*(\mathcal{L}). \tag{4.9}$$

Recall from equation (3.19) that all the divisor classes on X came as either pullbacks of classes on B or pullbacks of classes on B'. Thus, in general, $\mathcal{L}$ can be written as

$$\mathcal{L} = \pi'^* L \otimes \pi^* L', \tag{4.10}$$

where L and L' are global line bundles on B and B' respectively. The action of $\boldsymbol{FM}_X$ then splits into a Fourier-Mukai action on B and the trivial action on B'. Specifically

$$\boldsymbol{FM}_X(\mathcal{L}) = \pi'^* \boldsymbol{FM}_B(L) \otimes \pi^* L'. \tag{4.11}$$

Similarly, given the form (3.16) of τ_X, the action of $\boldsymbol{T}_X$ is given by

$$\boldsymbol{T}_X(\mathcal{L}) = \pi'^* \boldsymbol{T}_B(L) \otimes \pi^* \tau_{B'}^* L' \tag{4.12}$$

One notes that the action of $\boldsymbol{T}_X$ is simple on L' but more complicated on L. This reflects the fact that the Fourier–Mukai transformation is the action of T-duality on the π fibers.

From these relations it is straight-forward to deduce the action of $\boldsymbol{T}_X$ on the spectral data $(C, i_C^*(\mathcal{L}))$. The bundle is invariant under the involution provided that

$$\begin{aligned} \alpha_X(\Sigma) &= \Sigma, \\ \tau_{B'}^* L' &\cong L', \\ \boldsymbol{T}_B(L) &\cong L \end{aligned} \tag{4.13}$$

Finding solutions of the first two conditions it relatively easy. We can then use the general result (4.5) to try and solve the L condition. Rewriting the action of $\boldsymbol{T}_B$ in terms of cohomology, we see that (Prop. 2.11 in [21]).

$$c_1(\boldsymbol{T}_B(L)) = \tau_B^*(c_1(L)) + [c_1(L) \cdot (e - f)]\, f + [c_1(L) \cdot f + 1]\, (e - \zeta + f). \tag{4.14}$$

Using the action of τ_B^* given in Table 1, it is easy to see that $c_1(\boldsymbol{T}_B(L)) = c_1(L)$ if and only if $c_1(L)$ is in the affine subspace of $H^2(B, \mathbb{Q})$

$$-\frac{1}{2}e_1 + \mathrm{Span}(f, e_9, e_4 - e_5, e_4 - e_6, l - e_7 - 2e_8, 3l - 2(e_4 + e_5 + e_6) - 3e_7). \tag{4.15}$$

However, $c_1(L)$ must be in $H^2(B, \mathbb{Z})$ and there are no integral vectors in this subspace. Thus, we see that there are no solutions to the L condition.

We have derived a no-go theorem: V can never be τ_X-invariant if $\mathcal{N}_\Sigma$ is the pullback of a global line bundle on X. Instead we are forced to consider cases where $\mathcal{N}_\Sigma$ comes at least partly from additional classes on Σ.

4.2 Construction of V_r

Given the general results of the last section, we can now turn to the specific construction of a suitable family of rank r bundles V_r. We start by defining an appropriate spectral cover Σ. Again, we can use the projection π' to describe it as a pullback of a simpler object in B. Let C be a smooth irreducible curve in B suitable for a spectral construction in B. Specifically, let C be irreducible and in the divisor class

$$[C] = re + kf, \tag{4.16}$$

for some integer k, so that it is an r-fold cover of the base $\mathbb{P}^1$. We then take Σ to be the pullback

$$\Sigma = C \times_{\mathbb{P}^1} B'. \tag{4.17}$$

Given the properties of C, the spectral cover Σ will be a smooth, irreducible r-fold cover of B'. By construction, $\Sigma \to C$ is an elliptic fibration over C. In particular, it includes some number of reducible fibers. Let $f_1' = n_1' \cup o_1'$ and $f_2' = n_2' \cup o_2'$ be the reducible fibers of B' (as in equation (3.11)). Let F_1 and F_2 be the fibers of B over the corresponding points in $\mathbb{P}^1$. This is shown schematically in Figure 5. Given the way we glued the $\mathbb{P}^1$ bases of B and

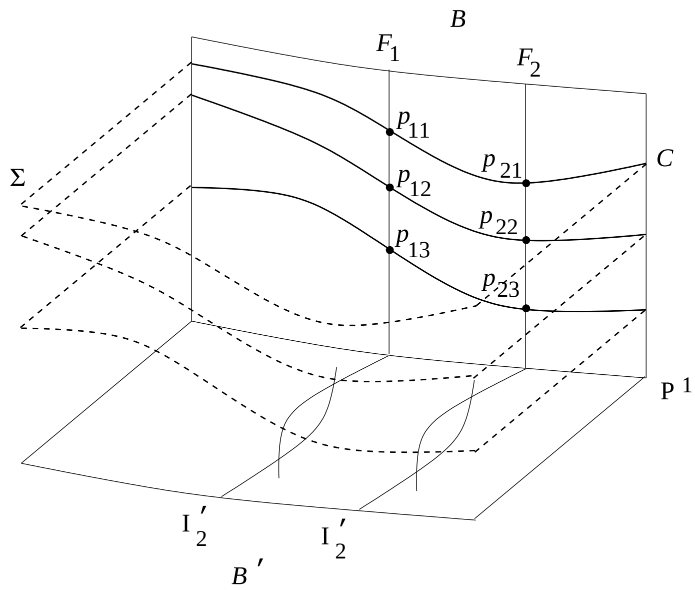

Figure 5: The structure of the spectral cover Σ.

B', the fibers F_1 and F_2 are smooth. The fibers F_1 and F_2 will each intersect the curve C in r points in B. Let us label these by an index $\kappa = 1, \ldots, r$, so

$$C \cap F_j = \{p_{j\kappa}\}_{\kappa=1}^{r} \tag{4.18}$$

for $j = 1, 2$. Above each of these points, the fiber of Σ will split. Thus Σ is an elliptic surface with $2r$ fibers of type I_2 given by $(n'_j \cup o'_j) \times \{p_{j\kappa}\}$ for $j = 1, 2$ and $\kappa = 1, \ldots, r$.

Next, we turn to the line bundle $\mathcal{N}_\Sigma$. Generically, there are three types of divisor on $i_\Sigma : \Sigma \hookrightarrow X$. First, there are pullbacks under i_Σ^* of global divisors on X. Then, we have pullbacks of divisors (points) on C and, finally, the $2r$ new divisors coming from the reducible fibers. Thus, we can take $\mathcal{N}_\Sigma$ of the form

$$\mathcal{N}_\Sigma = \pi'_{|\Sigma}{}^{*}\mathcal{N} \otimes \mathcal{O}_\Sigma \left(-\sum\nolimits_{j\kappa} \{p_{j\kappa}\} \times (a_{j\kappa} n'_j + b_{j\kappa} o'_j) \right) \otimes i_\Sigma^* \pi^* L, \tag{4.19}$$

where $\mathcal{N}$ is a line bundle of degree d on C, $a_{j\kappa}$ and $b_{j\kappa}$ are integers and L is a line bundle on B'. The first term is precisely the pullback of a bundle on C. The second is a bundle corresponding to some combination of the new divisors from the reducible fibers. The last term is the pullback of a global bundle on X. We note that there is some redundancy in the choices of $a_{j\kappa}$ and $b_{j\kappa}$. Since $n'_j + o'_j = f'_j$ is a pullback from $\mathbb{P}^1$, it can be absorbed in L. Thus, we are free to take

$$\text{either } a_{j\kappa} \geq 0,\ b_{j\kappa} = 0 \text{ or } a_{j\kappa} = 0,\ b_{j\kappa} \geq 0 \text{ for all } j \text{ and } \kappa. \tag{4.20}$$

Finally, we should stress that both the pullback of $\mathcal{N}$ and the bundles from the extra reducible fibers represent contributions to $\mathcal{N}_\Sigma$ which are not pullbacks of global line bundles

on X. Thus, we can hope to avoid the no-go theorem of section 4.1. In fact, generalizing to include $\mathcal{N}$ would be sufficient to find invariant bundles. However, as we will see, in order to satisfy conditions **(S)**–**(C3)**, one needs the additional freedom of the reducible fibers.

Having defined $(\Sigma, \mathcal{N}_\Sigma)$, we now need to understand the action of $\boldsymbol{FM}_X$ and $\boldsymbol{T}_X$. This could be addressed directly on X. However, in fact, the action can be decomposed in the following, relatively simple way. We first note that, as in equation (4.11) above, we can factor off the contribution of the global line bundle π^*L under the action of $\boldsymbol{FM}_X$. We have

$$V_r = \boldsymbol{FM}_X(i_{\Sigma *}\mathcal{N}_\Sigma) = \widetilde{W} \otimes \pi^* L, \tag{4.21}$$

where $\widetilde{W}$ is the bundle constructed from Σ and the spectral datum $\widetilde{\mathcal{N}}_\Sigma$ given by the first two terms in (4.19). If, somehow, we could also remove the contribution from the reducible fibers, we would then be left with a bundle which, given the structure of Σ, is just a pullback of a bundle W on B (similar to the case of equation (4.11)),

$$\boldsymbol{FM}_X(i_{\Sigma *}{\pi'_{|\Sigma}}^* \mathcal{N}) = {\pi'}^* W, \tag{4.22}$$

where,

$$W = \boldsymbol{FM}_B(i_{C*}\mathcal{N}). \tag{4.23}$$

This can then be calculated given our results on $\boldsymbol{FM}_B$ in section 4.1.

It turns out that there is a precise way to go from ${\pi'}^* W$ to $\widetilde{W}$. It is the action of a series of Hecke transforms. Details can be found in [21]. Here we will simply note that, given a vector bundle E on X, a divisor $i_D : D \hookrightarrow X$ and a short exact sequence $(\xi) : 0 \to F \to E_{|D} \to G \to 0$ of vector bundles on D, the associated Hecke transform generates a new vector bundle $\boldsymbol{Hecke}_{(\xi)}(E)$ on X. This new bundle has two characteristic properties: the Chern character of $\boldsymbol{Hecke}_{(\xi)}(E)$ is equal to $\mathrm{ch}(E) - \mathrm{ch}(i_{D*}G)$, and $\boldsymbol{Hecke}_{(\xi)}(E)$ is isomorphic to E on the complement of D.

$$\widetilde{W} = \boldsymbol{Hecke}_{a_{j\kappa}, b_{j\kappa}}({\pi'}^* W), \tag{4.24}$$

where $\boldsymbol{Hecke}_{a_{j\kappa}, b_{j\kappa}}$ represents $a_{j\kappa}$ successive Hecke transforms on the the divisor $D = F_j \times n'_j$ together with $b_{j\kappa}$ successive Hecke transforms on $D = F_j \times o'_j$.

In summary, we have built V_r as

$$V_r = \boldsymbol{Hecke}_{a_{j\kappa}, b_{j\kappa}}({\pi'}^*(\boldsymbol{FM}_B(i_{C*}\mathcal{N}))) \otimes \pi^* L. \tag{4.25}$$

In the remainder of this section we want to show that there are suitable $a_{j\kappa}$, $b_{j\kappa}$, L and $\mathcal{N}$ such that V_r is invariant under τ_X.

Acting with τ_X on V_r, it is clear from equation (4.21) and the form (3.16) of τ_X that

$$\tau_X^*(V_r) = \tau_X^*(\widetilde{W}) \otimes \pi^* \tau_{B'}^* L. \tag{4.26}$$

Thus, as in equation (4.13), invariance of V_r requires

$$\tau_{B'} L \cong L, \tag{4.27}$$

and

$$\tau_X^* \widetilde{W} \cong \widetilde{W}. \tag{4.28}$$

Recall that the Hecke transforms were on the divisors $F_j \times n'_j$ and $F_j \times o'_j$. From (3.13), we see that $\tau_{B'}$ exchanges n'_1 with o'_2 and n'_2 with o'_1. Hence, for invariance under τ_X we must perform the same number of Hecke transforms on each set of divisors paired under $\tau_{B'}$. This implies that $a_{1\kappa} = b_{2\kappa}$ and $a_{2\kappa} = b_{1\kappa}$. Given (4.20), we take

$$a_{1\kappa} = b_{2\kappa} \cong a_\kappa \geq 0, \qquad a_{2\kappa} = b_{1\kappa} = 0. \tag{4.29}$$

After undoing all the Hecke transforms, the τ_X-invariance of $\widetilde{W}$ reduces, from equation (4.22), to the τ_X-invariance of $\pi'^* W$. Since this is just the pull back of a bundle from B, given the expression (4.23), we finally have the condition

$$\boldsymbol{T}_B(i_{C*}\mathcal{N}) \cong i_{C*}\mathcal{N}. \tag{4.30}$$

Since $[C] = re + kf$, using the result (4.8) for global line bundles on B, one can show that equation (4.30) implies that

$$C = \alpha_B(C), \tag{4.31}$$

$$\mathcal{N} \cong \alpha^*_{B|C}(\mathcal{N}) \otimes \mathcal{O}_C(e - \zeta + f). \tag{4.32}$$

where $\alpha_{B|C}$ is the restriction of the involution α_B to C.

The requirement that $\tau_X^* V_r \cong V_r$, has been reduced to the four conditions (4.27), (4.29), (4.31) and (4.32). From Table 1, we see that there is a six-dimensional lattice of $\tau_{B'}$-invariant classes on B', so there are many possibilities for L. The conditions on $a_{j\kappa}$ and $b_{j\kappa}$ simply reduce to the choice of positive integers a_κ. Since the set of divisors C in the class $re + kf$ forms a projective space, there must be a solution to the condition on C. In general, C will be smooth provided

$$k \geq r > 1. \tag{4.33}$$

However, it is not obvious that the general C satisfying (4.31) will be smooth as well. In fact, for the values of k which we need, this turns out to be true for $r = 3$ and false for $r = 2$. The relevant curves C_2 are actually reducible and always have a vertical component. This requires a separate analysis of the invariance properties of W_2 which is carried out in [21]. In [21] it is also shown that for all bundles $\mathcal{N}$ on C there is some positive-dimensional torus of solutions to the condition (4.32).

We conclude, therefore, that using the construction (4.25), we can build a large family of τ_X-invariant rank-r bundles V_r on X.

4.3 Extensions and Stability

As we shall see in section 5, it turns out that even with the additional freedom coming from the reducible classes on Σ, the bundles V_r that we have just constructed are not quite general enough to satisfy all the conditions **(S)**–**(C3)**. We will actually construct our rank-five bundle V as the extension

$$0 \to V_2 \to V \to V_3 \to 0, \tag{4.34}$$

where V_r, with $r = 2, 3$, are τ_X-invariant rank-r bundles constructed via the procedure given above, and the extension class itself is also τ_X-invariant.

By construction, V will be τ_X-invariant. However, we also require **(S)** that V be stable given a suitable Kähler form H on X. This puts some constraint on the extension and also on the bundle V_2. Again, we will simply quote the result. Stability of V is equivalent, first, to the fact that V is not split, that is, it is not a direct sum,

$$V \neq V_2 \oplus V_3 \tag{4.35}$$

and, second, to a condition on the slopes

$$\mu(V_2) < \mu(V) \tag{4.36}$$

where, for an arbitrary bundle E, $\mu(E) = \left(\int_X c_1(E) \wedge H \wedge H\right) / \operatorname{rk}(E)$. (Note that condition **(C1)** implies that $\mu(V) = 0$.)

5. Numerical Conditions

In the previous section, we built a class of bundles V satisfying the τ_X-invariance condition **(I)**. We now need to find the requirements on V for simultaneously satisfying all the conditions **(S)**–**(C3)**. In particular, we will reduce them to a set of numerical constraints on the parameters defining V.

Recall that the invariance conditions (4.31) and (4.32) were geometrical in nature, fixing a particular C and $\mathcal{N}$. Let us assume these conditions are satisfied. Recalling that V is built from two bundles V_r, for $r = 2, 3$, the bundle is then determined by the following parameters, again indexed by r,

- the integers k_r giving the classes of the curves C_r as in (4.16),
- the integers $d_r = \deg(\mathcal{N}_r)$ giving the degrees of the bundles $\mathcal{N}_r$,
- the integers $a_{r\kappa}$ with $\kappa = 1, \ldots, r$ determining the number of Hecke transforms (4.29),
- and the line bundles L_r on B'.

The τ_X-invariance condition only constrains L_r. We have

(#I) $\tau_B^* L_r = L_r$, for $r = 2, 3$.

Recall that there was also a condition (4.33) in the construction of V_r in order for a generic C_r to be smooth. Here, it implies

$$k_2 \geq 2 \text{ and } k_3 \geq 3. \tag{5.1}$$

Let us now turn to the condition of stability **(S)**. We saw that this implied that V did not split (4.35) and a condition on the slopes (4.36). It can be shown [21] that these conditions amount to

(#Se) $L_2 \cdot f' > L_3 \cdot f'$,

(#Ss) $(2L_2 + (d_2 - 2k_2 + 1)f' - S_2^1(n_1' + o_2')) \cdot h' < 0$ for some ample class $h' \in H^2(\mathbb{Z}, B')$,

respectively. Here we have introduced the notation S_r^p for sums of p-th powers of $a_{r\kappa}$

$$S_r^p = \sum_{\kappa=1}^{r} (a_{r\kappa})^p \,. \tag{5.2}$$

What remains are the conditions **(C1)**–**(C3)** on the Chern classes. Using the explicit construction (4.25), one can derive the following expression for $\mathrm{ch}(V_r)$ for V_2 and V_3,

$$\begin{aligned}\mathrm{ch}(V_r) = r + \pi^* &\left(rL_r + \left(d_r - rk_r + \binom{r}{2} \right) f' - S_r^1(n_1' + o_2') \right) \\ &+ \left[\frac{r}{2} L_r^2 + \left(d_r - rk_r + \binom{r}{2} \right) L_r \cdot f' - S_r^1 \left(L_r \cdot n_1' + L_r \cdot o_2' \right) - 2S_r^2 \right] (f \times \mathrm{pt}) \\ &- k_r(\mathrm{pt} \times f') - k_r(L_r \cdot f')\mathrm{pt}.\end{aligned} \tag{5.3}$$

Note that $c_1(V_r)$ is a pullback from B', while the only terms in $\mathrm{ch}_2(V_r)$ are proportional to $f \times \mathrm{pt}$ and $\mathrm{pt} \times f'$. More significantly, one notes that requiring $c_1(V_r) = 0$ implies that $L_r \cdot f' = 0$ in B'. From the last term in (5.3) this, in turn, implies that $c_3(V_r)$ vanishes. Clearly, V_r by itself cannot, therefore, satisfy both conditions **(C1)** and **(C3)**. This is the reason we were forced to consider the generalization of V constructed as an extension.

Given the form of V, we have $\mathrm{ch}(V) = \mathrm{ch}(V_2) + \mathrm{ch}(V_3)$. Combining this with (5.3) and (3.20), the conditions **(C1)**–**(C3)** imply the following numerical constraints

(#C1) $2L_2 + 3L_3 = (S_2^1 + S_3^1)(n_1' + o_1') - (d_2 + d_3 - 2k_2 - 3k_3 + 4)f'$,

(#C2f) $k_2 + k_3 \leq 12$

(#C2f') $L_2^2 + \frac{3}{2}L_3^2 + (d_2 - 2k_2 + 1)(L_2 \cdot f') + (d_3 - 3k_3 + 3)(L_3 \cdot f') - (S_2^1 L_2 + S_3^1 L_3)(n_1' + o_2') - 2(S_2^2 + S_3^2) \geq -12$,

(#C3) $k_2(L_2 \cdot f') + k_3(L_3 \cdot f') = -6$.

Note that the $c_2(V)$ condition splits into two pieces, one from the component proportional to $f \times \mathrm{pt}$ and one from that proportional to $\mathrm{pt} \times f'$.

6. A Class of Solutions

What remains is to find a simultaneous solution of the equations **(#I)**, **(#Se)** and **(#Ss)**, and **(#C1)**–**(#C3)** together with the inequality (5.1). It is a straightforward, if tedious, procedure to calculate a fairly general solution [21]. Let us summarize the result, noting only that the main constraint on finding solutions is the tension between the stability condition **(#Se)** and the $c_2(V)$ conditions **(#C2f)** and **(#C2f')**.

First, solving conditions **(#C1)**, **(#C3)** and **(#I)**, constrains the line bundles L_r to have the following form

$$\begin{aligned} L_2 &= \frac{9}{k}\left(e' + \zeta'\right) + \frac{1}{2}\left(x - d_2 + 2k_2 - 1\right) f' + \frac{1}{2}\left(u + \frac{9}{k} + S_2^1\right)\left(n_1' + o_2'\right) + 3M \\ L_3 &= -\frac{6}{k}\left(e' + \zeta'\right) + \frac{1}{3}\left(-x - d_3 + +3k_3 - 3\right) f' + \frac{1}{3}\left(-u - \frac{9}{k} + S_3^1\right)\left(n_1' + o_2'\right) - 2M \end{aligned} \tag{6.1}$$

where $k = 2k_3 - 3k_2$ and x and u are as yet undetermined. From Table 1 and equation (3.13), we note that $e' + \zeta'$, f' and $n_1' + o_2'$ are τ_B-invariant classes. The parameter M represents an arbitrary τ_B invariant class orthogonal to $e' + \zeta'$, f' and $n_1' + o_2'$. It is then clear that the L_r satisfy **(#I)**.

Satisfying the inequalities **(#Se)**, **(#C2f)** and (5.1), and requiring that the L_r are integral classes, leaves only two possible values for k_2 and k_3,

$$\begin{gathered} k_2 = 3, \quad k_3 = 5, \quad \text{giving } k = 1 \\ \text{or} \\ k_2 = 3, \quad k_3 = 6, \quad \text{giving } k = 3 \end{gathered} \tag{6.2}$$

In general, M lies in a three-dimensional subspace spanned by

$$e_4' - e_5', \qquad e_4' - e_6', \qquad 3l' - 2e_4' - 2e_5' - 2e_6' - 3e_7'. \tag{6.3}$$

We will restrict ourselves to the one-dimensional subspace $M = z(e_4' - e_5')$ for some integer z. (Other solutions exist with more general M.)

Finally, we are left to satisfy the second stability condition **(#Ss)** and the inequality **(#C2f')**. It is straightforward to show that there is a solution

$$k_2 = 3, \qquad k_3 = 6, \tag{6.4}$$

together with

$$u = -3, \qquad z = 1, \qquad x = 5. \tag{6.5}$$

For the integers $a_{r\kappa}$ we have

$$a_{21} = a_{22} = a, \qquad b_{21} = b_{22} = b_{23} = b \tag{6.6}$$

for arbitrary non-negative integers a and b. Finally, the degrees d_2 and d_3 have the form

$$d_2 = 2p, \qquad d_3 = 3q + 1, \tag{6.7}$$

for arbitrary integers p and q.

In conclusion, we have constructed a large new class of bundles on non-simply connected Calabi-Yau manifolds which give three-family, anomaly-free vacua with the standard model gauge group. From equations (6.6) and (6.7), we see that rather than a single solution, we have a class of solutions depending on four arbitrary parameters. Furthermore, other solutions exist, with a more general class M in (6.1). This provides flexibility for discussing other physical properties of these models, such as nucleon decay and Yukawa couplings.

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